Proposing an algebra of intervalued fuzzy sets

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ABSTRACT

The intersection, union and negation of intervalued fuzzy sets are defined using generalized operators of t-norms and t-conorms. The properties of different intervalued logics are studied. Some examples are given.

***Keywords*:**
Fuzzy Logic, intervalued fuzzy sets, álgebra, t-norm

1. INTRODUCTION

The intervalued fuzzy sets on a universe X express a characteristic of the elements of a universe X by an interval in [0, 1]x[0, 1].

Sometimes it is more useful to an expert to assign a confidence interval to a characteristic than a membership degree. For example, it can be easier to say that a characteristic is satisfied in a range between 0.7 and 0.9, than to say that it has membership degree of 0.82.

If we have two or many experts expressing their knowledge with intervalued fuzzy sets, we must aggregate their knowledge, so we need to define some operations on intervalued fuzzy sets.

In this paper we propose a definition of conjunction, disjunction and negation of intervalued fuzzy sets.

2. PRELIMINARIES

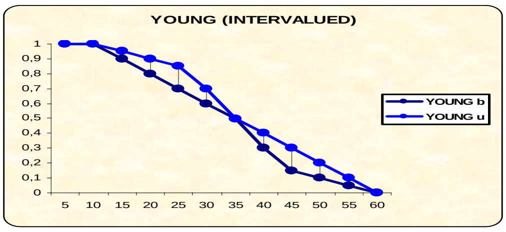
Definition 2.1:

Let X be a universe and let A be a characteristic (or linguistic variable) on X. An intervalued fuzzy set is a mapping A: X → [0, 1] x [0, 1].

Note that A can be also defined by two fuzzy sets A_U and A_B on X, with $A_U \supseteq A_B$, denoting the characteristic 'upper axis' and 'bottom axis' of the characteristic A given for every element in the universe of discourse X. Therefore $A(x) = [A_B(x), A_U(x)]$.

Example 2.1:

Let X be a set of ages (in years) and let consider the characteristics young and adult on X. An intervalued fuzzy set YOUNG and ADULT are defined in Figure 1 as:



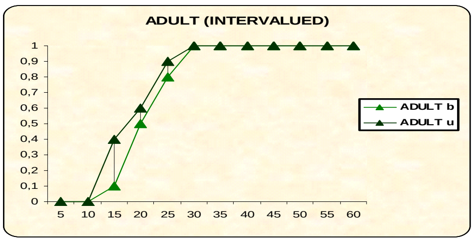


Figure 1: Intervalued fuzzy set YOUNG and ADULT

The intervalued fuzzy sets generalize the fuzzy sets, in the case that $A_U(x) = A_B(x) \ \forall \ x \in X$, therefore also generalize crisp sets.

3. CONCEPTS ON INTERVALUED SUBSETS

The following definitions for intervalued fuzzy sets generalize those on fuzzy sets theory.

Let A be an intervalued set on X with upper axis A_U and bottom axis A_B.

Definition 3.1. Normal Intervalued Subset.

A is normal intervalued subset if $\exists \ x \in X \ / \ A_U(x) = A_B(x) = 1$.

Definition 3.2. Level Intervalued α-cut

The α-cut of A is the crisp set $A_\alpha = \{x: A_U(x) \geq A_B(x) \geq \alpha\} \ \forall \ \alpha$ in [0, 1]

Definition 3.3. Intervalued Subset

A is an intervalued subset of B ($A \subseteq B$) if and only if: $[A_U(x) \leq B_U(x)$ and $A_B(x) \leq B_B(x)] \ \forall \ x \in X$

Definition 3.4. Equality

A is equal to B if and only if $A_U(x) = B_U(x)$ and $A_B(x) = B_B(x) \ \forall \ x \in X$

4. GENERALIZED OPERATORS

4.1 Generalized t-norms

Definition 4.1.1

A generalized t-norm is an operator TI: $[[0, 1]x[0, 1]] \ x \ [[0, 1]x[0, 1]] \rightarrow [[0, 1]x[0, 1]]$ that satisfying the following properties:

Border condition: $TI([x_B, x_U], [0, 0]) = [0, 0] \quad \forall \ x_B, x_U \in [0, 1]$

$TI([x_B, x_U], [1, 1]) = [x_B, x_U] \quad \forall \ x_B, x_U \in [0, 1]$

Commutative: $TI([x_B, x_U], [y_B, y_U]) = TI([y_B, y_U], [x_B, x_U])$

Monotonous: If $x_B \geq x'_B$ and $x_U \geq x'_U$ and $y_B \geq y'_B$ and $y_U \geq y'_U$ then $TI([x_B, x_U], [y_B, y_U]) \geq TI([x'_B, x'_U], [y'_B, y'_U])$

Associative: $TI([x_B, x_U], TI([y_B, y_U], [z_B, z_U])) = TI(TI([x_B, x_U], [y_B, y_U]), [z_B, z_U])\ \forall\ x_B, x_U, y_B, y_U, z_B, z_U \in [0, 1]$

Definition 4.1.2

Let T_B, T_U be triangular norms [8]. The $TI_{Tb,Tu}$ operator is defined by:

$$TI_{Tb,Tu}([x_B, x_U], [y_B, y_U]) = [T_B(x_B, y_B), T_U(x_U, y_U)]$$

Corollary 4.1 $TI_{Tb,Tu}$ is a generalized t-norm. The proof is trivial by the t-norms axioms [8].

Example 4.1

$$TI_{min,min}([x_B, x_U], [y_B, y_U]) = [\min(x_B, y_B), \min(x_U, y_U)]$$

4.2 Intersection of intervalued fuzzy sets

Definition 4.2.1

Let T_B, T_U be the t-norms. The T_B,T_U-intersection of two intervalued fuzzy sets A and B is an intervalued fuzzy set

$(A \cap_{Tb,Tu} B): X \rightarrow [0, 1] \times [0, 1]$ such that:

$$(A \cap_{Tb,Tu} B)(x) = TI([A_B(x), A_U(x)], [B_B(x), B_U(x)])$$

$$= [T_B(A_B(x), B_B(x)), T_U(A_U(x), B_U(x))]\ \forall\ x \in X$$

Note: For the upper axis we will apply any t-norm (Min, Prod, W, etc.), which we call T_U. But for the bottom axis we must apply a t-norm T_B, which must satisfy the following condition: $T_U(x,y) \geq T_B(x,y)\ \forall\ x,y \in X$

Example 4.2.1: ***Min,min-intersection of two intervalued fuzzy sets YOUNG and ADULT is computed in Figure 2.***

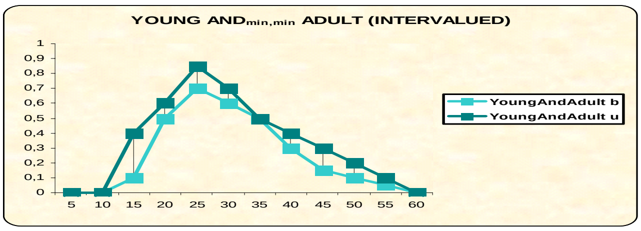


Figure 2: $\cap$min,min conjuntion of intervalued fuzzy sets YOUNG and ADULT.

Definition 4.2.1

A generalized t-conorm is an operator SI: $[[0, 1]x[0, 1]] \times [[0, 1]x[0, 1]] \rightarrow [[0, 1], [0, 1]]$ satisfying the following properties:

Border condition: $SI([x_B, x_U], [0, 0]) = [x_B, x_U]\ \ \forall\ x_B, x_U \in [0, 1]$

$$SI([x_B, x_U], [1, 1]) = [1, 1]\ \ \forall\ x_B, x_U \in [0, 1]$$

Commutative: $SI([x_B, x_U], [y_B, y_U]) = SI([y_B, y_U], [x_B, x_U])$

Monotonous: If $x_B \geq x'_B$ and $x_U \geq x'_U$ and $y_B \geq y'_B$ and $y_U \geq y'_U$ then $SI([x_B, x_U], [y_B, y_U]) \geq SI([x'_B, x'_U], [y'_B, y'_U])$

Associative: $SI([x_B, x_U], SI([y_B, y_U], [z_B, z_U]))= SI(SI([x_B, x_U], [y_B, y_U]), [z_B, z_U]) \; \forall \; x_B, x_U, y_B, y_U, z_B, z_U \in [0, 1]$

Definition 4.2.2
Let S_B, S_U be triangular conorms [8].

$SI_{Sb,Su}$ is an operator defined by

$SI_{Sb,Su}([x_B, x_U], [y_B, y_U]) = [S_B(x_B, y_B) , S_U(x_U, y_U)]$

Example 3.2.1:

$SI_{max,max}([x_B, x_U], [y_B, y_U]) = [max(x_B, y_B) , max(x_U, y_U)]$

4.3 Union of intervalued fuzzy sets

Definition 4.3.1
Let A be an intervalued set on X with upper axis A_U and bottom axis A_B.

Let B be an intervalued set on X with upper axis B_U and bottom axis B_B.

Let S_B be the t-conorm used to compute the union of the bottom axes.

Let S_U be the t-conorm used to compute the union of the upper axes.

The S_B,SU-union of two intervalued fuzzy sets A and B is a intervalued fuzzy set defined as follows

$(A \cup_{Sb,Su} B): X \rightarrow [0, 1] \times [0, 1]$ such that:

$$(A \cup_{Sb,Su} B)(x) = SI([A_B(x), A_U(x)], [B_B(x), B_U(x)])$$
$$= [S_B(A_B(x), B_B(x)), S_U(A_U(x), B_U(x))] \; \forall \; x \in X$$

Note: For the bottom axis we will apply any t-conorm (Max, Prod*, W*, etc.), which we will call S_B.

But for the upper axis we must apply a t-conorm, which we will call S_U, which must meet the following condition: $S_U(x, y) \geq S_B(x, y) \; \forall \; x, y \in X$

Example 4.3.1*:*

Max,max-union of two intervalued fuzzy sets YOUNG and ADULT is computed in Figure 3.

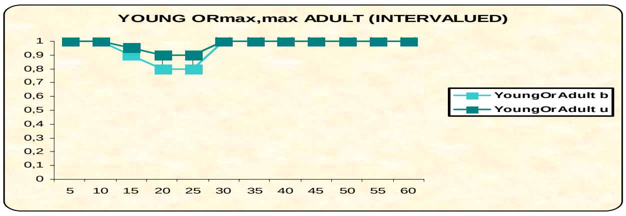


Figure 3: ∪max,max of intervalued fuzzy sets YOUNG and ADULT.

4.4 Generalized negation operator

Definition 4.4.1

The operator NI: [[0, 1]x[0, 1]] → [[0, 1]x[0, 1]] is a generalized strong negation if it satisfies the following axioms:

Border condition: NI([0, 0]) = [1, 1], NI([1, 1]) = [0, 0].

Monotonous: If $x_B \leq y_B$ and $x_U \leq Y_U$ then $NI([x_B, x_U]) \geq NI([y_B, y_U])$ $\forall x_B, x_U, y_B, y_U \in [0, 1]$

NI is continuous.

Involution: $NI(NI([x_B, x_U])) = [x_B, x_U]$ $\forall x_B, x_U \in [0, 1]$

Definition 4.4.2

Let N_B, N_U be negation operators [7]. The generalized negation operator is an operator $NI_{Nb,Nu}$ defined by

$NI_{Nb,Nu}([x_B,x_U]) = [N_U(x_U), N_B(x_B)]$

Corollary:

$NI_{Nb,Nu}$ is a generalized strong negation. The proof is trivial by the strong negation axioms [7].

Example:

$NI_{1-x,1-x}([x_B, x_U]) = [1-x_U, 1-x_B]$

4.5 Negation of intervalued fuzzy sets

Definition 4.5.1

Let A be an intervalued set on X with upper axis A_U and bottom axis A_B.

Let N_B be the negation operator used to compute the negation of the bottom axes and let N_U be the negation operator used to compute the negation of the upper axes.

The N_B,N_U-negation of an intervalued fuzzy set A is a intervalued fuzzy set $A'_{Nb,Nu}$: X → [0, 1] x [0, 1] defined as follows:

$$A'_{Nb,Nu}(x) = NI_{Nb,Nu}([A_B(x), A_U(x)]) = [N_U(A_U(x)), N_B(A_B(x))] \ \forall x \in X$$

Example 3.6.1:
(1-x),(1-x)-negation of the intervalued fuzzy set YOUN is computed in Figure 4.

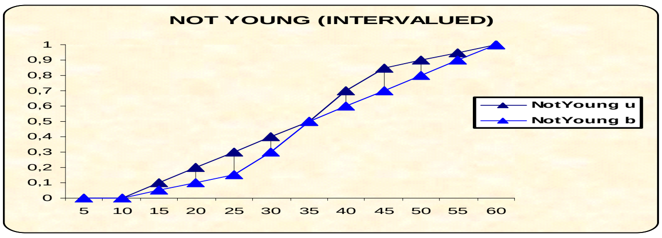


Figure 4: Intervalued fuzzy set NOT YOUNG

4.6 Properties of Intervalued logics (TI, SI, NI)

The ([min, min], [max, max], [1 - x, 1 - x]) intervalued logic does not verify the Contradiction Law: $(A \cap_{min,min} A'_{1-x,1-x}) = [0, 0]$ and Excluded Third Law: $(A \cup_{max,max} A'_{1-x,1-x}) = [1, 1]$. However this logic fits the idempotence property, and therefore this logic verifies the distributive property. It is a Morgan lattice. The ([Prod, Prod], [Prod*, Prod*] , [1-x, 1-x]) intervalued logic does not satisfy the distributive property (Prod is arquimedian) and does not verify the contradiction law, nor the excluded third law.

The ([W, W], [W*, W*], [1-x, 1-x])intervalued logic, where W, W* are the Lukasiewicz t-norm and t-conorm, does not verify the distributive property, because W is an arquimedian t-norm. However it verifies the contradiction and the excluded third law.

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